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THE HYDRODYNAMIC FIELD DUE TO FORCING BY MODULATED ACOUSTIC WAVES

M. J. LYELL

Mechanical and Aerospace Engineering Department, Box 6106, West Virginia University, Morgantown, WV 26506-6106, U.S.A.

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Abstract—This theoretical investigation determines the hydrodynamic field interior and exterior to an oscillating viscous drop for the case in which the hydrodynamic field exists strictly as a result of forcing by a modulated acoustic standing wave field. Perturbation of the governing equations of fluid motion generates a hierarchy of systems of equations. At second order, this system represents the hydrodynamic field which is forced by terms quadratic in acoustic field quantities. Viscous effects are incorporated into the acoustic field. Of particular interest is the nature of the hydrodynamic field in a sublayer region which exists near the drop–host interface. The boundary conditions, which are imposed at the drop–host fluid interface and which include the radial and tangential forcing due to the acoustic radiation pressure vector, complete the problem formulation.

Key Words: drop dynamics, multiphase flow, acoustic forcing, acoustic levitation

1. INTRODUCTION

This work investigates the flow field of an oscillating liquid drop which is surrounded by a viscous host medium for the particular case in which the drop oscillations occur strictly as a result of acoustic forcing. This problem arises naturally in the context of acoustic levitation systems which have been utilized not only in ground-based experiments, but also in investigations done in the microgravity environment present in the space shuttle.

The development of acoustic levitation systems has provided a technology which can be utilized for fundamental liquid droplet studies. For example, drop oscillations as well as applications to emulsification and splitting of drops have been studied by Marston (1980). Acoustic levitation devices utilize radiation pressure forces to position the fluid sample (drop) away from container walls. For the case of the three-axes system discussed in Wang *et al.* (1984), acoustic drivers (speakers) centered in three orthogonal sides of a parallelepiped chamber are driven at its resonant frequency. A standing wave pattern is then set up, and liquid drops can be positioned in the region in which the pressure is a minimum, i.e. the wave pressure nodes. The drop oscillations can then be induced via frequency modulation of an acoustic wave.

The quadrupole resonance of a simple drop has been investigated experimentally by Marston & Apfel (1980). Drop size was on the order of millimeters. Modulated acoustic radiation pressure provided the driving force. Small amplitude oscillations and decay of a free (non-driven) drop were studied experimentally by Trinh *et al.* (1982). Furthermore, large-amplitude drop shape oscillations have been investigated experimentally for both the free and forced cases by Trinh & Wang (1982). Drop oscillations and break-up were studied in a visualization experiment by Marston & Goosby (1985). The aforementioned experiments were all performed on Earth in a 1 g field. The drop itself was surrounded by an immiscible host fluid, with positioning in the chamber accomplished via acoustic radiation forces. It was deduced from experiments that the effects from tangential acoustic radiation pressure forcing appeared less significant than those due to forcing in the radial direction.

A theoretical analysis of the hydrodynamic flow field resulting from acoustic forcing, both within the drop and the host fluids, accompanied early experimental work by Marston (1980). In the analysis, the acoustic field was assumed *a priori* to be irrotational. With this assumption, any tangential radiation pressure forcing at the interface separating the drop and the host fluid is set identically to zero. Such forcing normally would enter the analysis through the boundary conditions. More significantly, the assumption of irrotationality of the acoustic field precludes any forcing of the Navier–Stokes equation (which governs the hydrodynamic field). Since the hydrodynamic field exists only as a result of the acoustic forcing, such source terms due to acoustic forces are expected.

The present work does not make the restrictive assumption on the acoustic field. The goal of this theoretical effort is to determine the hydrodynamic field, both interior and exterior to the drop, as the result of acoustic forcing from first principles. In this work, a consistent perturbation expansion scheme in a small parameter generates a hierarchy of systems of equations. The lowest order set of equations governs the acoustic field, as shown by Lyell (1993). The next order represents the hydrodynamic problem; in it the acoustic field naturally couples into the governing equations. These equations, together with the associated boundary conditions, are solved. The results are compared with those found in the early simplified analysis of Marston (1980), and the significance of the tangential forcing is addressed.

For the reader who is interested in pursuing the literature in great detail, please note that there are a number of misprints in the work of Marston (1980). These misprints, which do not interfere with the conceptual flow of that work, have been corrected in a later paper by Marston *et al.* (1982). As a further point, please note that the determination of the hydrodynamic field in Marston (1980) was cast in terms of the radial components of the velocity and vorticity fields. In the current work, the presentation is more straightforward, with the formulation cast in terms of the pressure and velocity field components. Because of length, intermediate steps in equation development cannot be presented here. These steps are detailed in appendices I–V, a copy of which may be obtained from the Editor.

2. FORMULATION AND EQUATION DEVELOPMENT

The goal is to generate the equations governing the hydrodynamic field, with the forcing due to the viscous acoustic field arising and coupling in naturally. As the hydrodynamic field of the drop would not exist were it not for the acoustic field, it is to be expected that the hydrodynamic field will arise at higher order in the expansion scheme. It is therefore necessary to introduce elements of the acoustic field problem. These are kept to the minimum necessary for logical exposition of the current work, since details on the nature of the viscous acoustic field have been presented by Lyell (1993). Emphasis is on elucidation of the connection between the hydrodynamic and acoustic fields.

The acoustic field is modeled as a symmetric, modulated standing wave field, with the acoustic waves represented as plane waves. The walls of the levitator are taken to be far from the drop, and the drop is not influenced by them. The center of mass of the drop is stationary. Focus is on the induced drop oscillation; the acoustic wave used to position the drop is not taken into account in the analysis. For the terrestrial environment, the effect of this positioning wave in the deformation of the drop will be negligible provided the drop radius is smaller than the square root of the ratio of interfacial surface tension to buoyancy force (for unit volume). This is taken to be the case.

Both acoustic and hydrodynamic fields must satisfy the governing equations of fluid mechanics: continuity, the Navier–Stokes equations (only Newtonian fluids considered) and the conservation of energy. The analysis is isothermal, which results in simplifications of the energy equation. Material properties such as viscosity are taken to be constant. In general, these values differ for drop and host fluids. The analysis is restricted to axisymmetric.

Let the field variables be non-dimensionalized with respect to quantities which reflect scales in the acoustic field problem. Quantities which pertain to the host (drop) region are denoted by a superscript o(i). Dimensionless quantities are denoted by a tilde.

$$(c_o^o/\omega_{AC})\tilde{\mathbf{x}} = \mathbf{x}$$

$$(\omega_{AC}^{-1})\tilde{t} = t; \quad (\rho_o^o)\tilde{\rho} = \rho$$

$$(c_o^o)\tilde{\mathbf{u}} = \mathbf{u}; \quad \rho_o^o(c_o^o)^2\tilde{p} = p \qquad [1]$$

with c°_{o} the speed of sound in the host medium and ω_{AC} the acoustic frequency. The density of the undisturbed host medium is used as the reference density. It is of crucial importance that the length scale of the hydrodynamic problem, denoted by d, be equal to that in the acoustic problem. If the field variables were to be non-dimensionalized by scales pertinent to the hydrodynamic field, then the non-dimensional quantities (indicated by a caret) would be given by

$$d\hat{\mathbf{x}} = \mathbf{x}; \quad d = (c^{\circ}_{\mathrm{o}})/\omega_{\mathrm{AC}}$$
$$\omega_{\mathrm{D}}^{-1}\hat{t} = t; \quad \omega_{\mathrm{D}} d\hat{\mathbf{u}} = \mathbf{u}$$
$$\rho^{\circ}_{\mathrm{o}} d^2 \omega_{\mathrm{D}}^2 \hat{p} = p \qquad [2]$$

with ω_{D} the natural frequency of oscillation of an inviscid drop of the same dimensions and having the same interfacial tension at the drop–host interface. A comparison of the non-dimensional quantities shows that

$$\tilde{\mathbf{u}} = \delta\hat{\mathbf{u}}; \quad \tilde{p} = \delta^2 \hat{p} \qquad [3]$$

with δ the ratio of the acoustic and hydrodynamic time scales, i.e. $(\omega_{\mathrm{D}}/\omega_{\mathrm{AC}})$

Utilizing scales relevant to the acoustic field, expand the non-dimensionalized variables in the host region in powers of δ as follows

$$\tilde{p}^{\circ} = \tilde{p}^{\circ}_{\mathrm{m}} + \delta\tilde{p}^{\circ}_{\mathrm{AC}} + \delta^2\tilde{p}^{\circ}_{\mathrm{HY}}$$
$$\tilde{\rho}^{\circ} = 1 + \delta\tilde{\rho}^{\circ}_{\mathrm{AC}}$$
$$\tilde{\mathbf{U}}^{\circ} = \delta\tilde{\mathbf{v}}^{\circ}_{\mathrm{AC}} + \delta^2\tilde{\mathbf{u}}^{\circ}_{\mathrm{HY}} \qquad [4]$$

The subscripts AC and HY indicate the acoustic and hydrodynamic field quantities, respectively. Except for the replacement of "1" in the density expansion by $\beta = \rho^{\mathrm{i}}_{\mathrm{o}}/\rho^{\circ}_{\mathrm{o}}$, the expansion of the field variables in the drop region is the same.

The expansions are substituted into the governing equations. At order δ, this results in sets of equations (for host and drop regions) which the acoustic field must satisfy. The acoustic field has been determined; in particular, the viscous correction was determined via the method of composite expansions. At this stage, the acoustic field quantities are regarded as known. See Lyell (1993) (or appendix I for an outline).

Before proceeding with the investigation of the hydrodynamic field, several remarks concerning the acoustic field are in order. It is frequency modulated, and can be viewed as the superposition of two acoustic standing waves, the first of which has the frequency ω_{AC}. The second is of frequency ω''. Moreover, $\omega_{\mathrm{AC}} = \omega'' - \omega_{\mathrm{D}}$, with $\omega_{\mathrm{D}} \ll \omega_{\mathrm{AC}}$ and $\omega_{\mathrm{D}} \ll \omega''$. The drop frequency, much smaller than those of the acoustic waves, is akin to a beat frequency. The respective amplitudes of the acoustic waves are taken to be the same.

The hydrodynamic field which occurs at second order in δ is solenoidal. At this order, the time average of the equations over an acoustic period is taken. The resulting equations are time dependent, with the time dependence being of the form $\cos(\delta t + \eta'' - \eta')$. The term $(\eta'' - \eta')$ represents the phase. The temporal behavior in the hydrodynamic problem is not on the same scale as in the acoustic problem. It is convenient to define a new independent variable for time. Let $T = \delta t$, with T a slow time. Moreover, the dependent variables are to be re-expressed utilizing this hydrodynamic scale time, where appropriate. This results in the rescaling introduced in [3]. The resulting hydrodynamic equations are, in the host region,

$$\nabla \cdot \hat{\mathbf{u}}^{\circ}_{\mathrm{HY}} = 0$$
$$\partial\hat{\mathbf{u}}^{\circ}_{\mathrm{HY}}/\partial T + \nabla\hat{p}^{\circ}_{\mathrm{HY}} - (1/\mathrm{Re}_{\mathrm{HY}})\nabla^2\hat{\mathbf{u}}^{\circ}_{\mathrm{HY}} = -2((\bar{\mathbf{v}}^{\circ}_{\mathrm{AC}}) \cdot \nabla(\bar{\mathbf{v}}^{\circ}_{\mathrm{AC}})^* + (\bar{\rho}^{\circ}_{\mathrm{AC}})(\mathrm{i}\bar{\mathbf{v}}^{\circ}_{\mathrm{AC}}) + CC)\cos(T + \eta'' - \eta') \qquad [5]$$

The Reynolds number $\mathrm{Re}_{\mathrm{HY}} = d^2\omega_{\mathrm{D}}/\nu^{\circ}_{\mathrm{o}}$ is expressed in terms of quantities relevant in the hydrodynamic problem. It is noted that $\mathrm{Re}_{\mathrm{AC}} = \mathrm{Re}_{\mathrm{HY}}/\delta$ and $\mathrm{Re}_{\mathrm{HY}}$ is taken to be an order one quantity. The overbar indicates the time independent part, and the asterisk and "CC" indicate the complex conjugate. In the drop region, the hydrodynamic equations are

$$\nabla \cdot \hat{\mathbf{u}}^{i}_{\mathrm{HY}} = 0$$

$$\beta\, \partial \hat{\mathbf{u}}^{i}_{\mathrm{HY}}/\partial T + \nabla \hat{p}^{i}_{\mathrm{HY}} - (\alpha/\mathrm{Re}_{\mathrm{HY}})\nabla^{2}\hat{\mathbf{u}}^{i}_{\mathrm{HY}}$$
$$= -2(\beta(\bar{\hat{\mathbf{v}}}^{i}_{\mathrm{AC}}) \cdot \nabla(\bar{\hat{\mathbf{v}}}^{i}_{\mathrm{AC}})^{*} + (\bar{\hat{\rho}}^{i}_{\mathrm{AC}})(\mathrm{i}\bar{\hat{\mathbf{v}}}^{i}_{\mathrm{AC}})^{*} + \mathrm{CC})\cos(T + \eta'' - \eta') \quad [6]$$

The parameters α and β are μ^{i}_{o}/μ^{o}_{o} and $\rho^{i}_{o}/\rho^{o}_{o}$, respectively.

It is [5] and [6], together with the appropriate boundary–interface conditions, which must be solved in order to determine the hydrodynamic field. The forcing terms on the right-hand sides of the conservation of momentum equations in drop and host fluids involve acoustic field quantities, and are known functions. The temporal cosine term is replaced by one in exponential form. At this stage, the caret is dropped for convenience.

It is noted that the time averaging process also yields a time independent set of equations; the solutions to which are termed the static solution. As it is the drop oscillation problem which is of primary interest, comments on the time independent problem are relegated to the discussion (and appendix III, from the Editor).

3. HYDRODYNAMIC FIELD SOLUTION

The acoustic field solutions act as forcing terms in [5] and [6]. A solution of the acoustic field quantities in the drop and host regions has been found by Lyell (1993). Brief details are reprised in appendix I (available from the Editor). The effects of viscosity were found to be important in regions of dimension $\sqrt{(\nu^{o}_{o}/\omega_{\mathrm{AC}})}$ extending both inward and outward from the drop–host interface. In this region, the spatial independent variable in the radial direction, r, was rescaled. In the host (drop) region, $r = R + \epsilon\zeta$ ($r = R - \epsilon\xi$). The small parameter ϵ equals $\sqrt{(1/\mathrm{Re}_{\mathrm{AC}})}$. The viscosity modifies the acoustic solution only in the sublayer region; outside of it the acoustic field is strictly inviscid and irrotational.

Suppress the temporal dependence and let

$$\mathbf{v}^{\mathrm{i,o}}_{\mathrm{AC}} = \mathbf{v}^{\mathrm{i,o}}_{\mathrm{AC}}(r, \theta) + \mathbf{v}^{\mathrm{i,o}}_{\mathrm{AC}}(\xi\ (\text{or } \zeta), \theta) \quad [7]$$

The first term on the right-hand side denotes the solution which exists outside the sublayer region in the drop (i) or host (o). It is inviscid and irrotational. The second term on the r.h.s. exists as a result of viscous effects on the acoustic field, and decays to zero outside the sublayer regions in the drop and host fluids.

If one manipulates the conservation of momentum equation for the hydrodynamic field (in drop and host regions) by taking the curl of [5] and [6], it will be found that there is no contribution from terms which have no ζ or ξ dependence. It is in the sublayer regions of the drop and host fluids that the hydrodynamic vorticity equations are forced by acoustic field quantities which act as sources of vorticity.

With the structure of the acoustic field in mind, decompose the hydrodynamic field quantities as follows

$$\mathbf{u}^{\mathrm{io}}_{\mathrm{HY}}(r, \theta) = (\bar{\bar{\mathbf{u}}}^{\mathrm{i,o}}_{\mathrm{HY}}(r, \theta) + \hat{\bar{\mathbf{u}}}^{\mathrm{i,o}}_{\mathrm{HY}}(\xi (\text{or } \zeta), \theta))\, \mathrm{e}^{(i(T + \eta_{\mathrm{RSP,1}}))}$$

$$p^{\mathrm{i,o}}_{\mathrm{HY}}(r, \theta = (\bar{\bar{p}}^{\mathrm{i,o}}_{\mathrm{HY}}(r, \theta) + \hat{\bar{p}}^{\mathrm{i,o}}_{\mathrm{HY}}(\xi (\text{or } \zeta), \theta))\, \mathrm{e}^{(i(T + \eta_{\mathrm{RSP,1}}))}$$

with $\eta_{\mathrm{RSP,1}}$ the phase associated with the hydrodynamic field. The exponential time-dependent terms will be denoted by EXP. This decomposition is substituted into the conservation of mass and momentum equations [5] and [6].

The discussion which follows is done in terms of the host fluid region. As the development in the drop region proceeds in a similar manner, results are only presented in that region. It is noted that the two sub-regions are delineated, an inner region in which source terms for vorticity appear, and an outer region in which the hydrodynamic vorticity field is unforced.

Inner host fluid region

In this region, recall that the independent variable r was stretched, with $r = R + \epsilon\zeta$, and that $\epsilon = \sqrt{(1/\mathrm{Re}_{\mathrm{AC}})}$. Utilizing the relationship between $\mathrm{Re}_{\mathrm{AC}}$ and $\mathrm{Re}_{\mathrm{HY}}$, ϵ can be re-expressed as $\epsilon = \sqrt{(\delta/\mathrm{Re}_{\mathrm{HY}})}$. Recall that $\mathrm{Re}_{\mathrm{HY}}$ is an order one quantity. The quantities originating from the acoustic field which appear on the r.h.s. of the conservation of momentum equation are written in terms of the ζ dependence. Then, $\mathbf{u}_{\mathrm{HY}}$ and p_{HY} are expanded in terms of $\sqrt{(\delta/\mathrm{Re}_{\mathrm{HY}})}$. The same process is applied to the conservation of mass equation. To lowest order, the forcing of the hydrodynamic field by acoustic field quantities is given by the equation

$$\partial(\hat{\#}\mathrm{u}^{\circ}_{\theta 1,\mathrm{HY}})/\partial\zeta^2 = (2\mathrm{Re}_{\mathrm{HY}}) \times (\mathrm{TERM1}) \times (\mathrm{TERM2}) \times (-(1-i)/2^{1/2}\exp((-(1-i)/(2^{1/2}))\zeta) + \mathrm{CC}) \times \mathrm{e}^{(i(\eta''-\eta'-\eta_{\mathrm{RSP,l}}))} \quad [8]$$

with

$$\mathrm{TERM1} = \sum_{l'} (q_{l'}, B^{\circ}_{\mathrm{BL},l'}(\mathrm{d}P_{l'}/\mathrm{d}\theta))^*$$

and

$$\mathrm{TERM2} = \sum_{l''} ((\alpha^{\circ}_{\mathrm{s},l''} q_{l''}(h^1_{l''}(r))'|_{\mathrm{R}} + A_{\mathrm{NC}} q_{l''}(j_{l''}(r))'|_{\mathrm{R}} P_{l''})$$

where the P_l are Legendre polynomials. Again, the hydrodynamic field of the host fluid region expanded in the inner variable, ζ, is indicated by $\hat{\#}$. Coefficients which appear in TERM1 and TERM2 are $B^{\circ}_{\mathrm{BL},l}$, $\alpha^{\circ}_{\mathrm{s},l}$, q_l and A_{NC}. These occur in the solution of the acoustic field, and are known quantities in the hydrodynamic field problem. The subscript 1 on the $\hat{e}_\theta$ component of velocity in [8] indicates the contribution is at order $\sqrt{(\delta/\mathrm{Re}_{\mathrm{HY}})}$ in the expansion scheme. The solution is given by

$$\hat{\#}\mathrm{u}^{\circ}_{\theta 1,\mathrm{HY}} = \sum_{l} (\mathrm{d}P_l/\mathrm{d}\theta)(T1 + T2 + T3)\exp((-(1+i)/2^{1/2})\zeta)\mathrm{e}^{i(\eta''-\eta'-\eta_{\mathrm{RSP,l}})} \quad [9]$$

with

$$T1 = \mathrm{Re}_{\mathrm{HY}}(-1)2^{1/2}(1+i)(C1^{\circ}_l)\exp((-(1-i)/(2)^{1/2})\zeta)$$

$$T2 = \mathrm{Re}_{\mathrm{HY}}(-1)2^{1/2}(1-i)(C21^{\circ}_l)\exp((-(1+i)/(2)^{1/2})\zeta)$$

and

$$T3 = d^{\circ}_0 + d^{\circ}_1(\zeta)$$

and with

$$C1^{\circ}_l = ((2l+1)/2l(l+1))\int_0^{\pi}\left(\sum_{\mathrm{L}} C^{\circ}_{\mathrm{L}}(\mathrm{d}P_{\mathrm{L}}/\mathrm{d}\theta)\right)(\mathrm{d}P_l/\mathrm{d}\theta)\sin(\theta)\,\mathrm{d}\theta$$

$$C2^{\circ}_l = ((2l+1)/2l(l+1))\int_0^{\pi}\left(\sum_{\mathrm{L}} (C^{\circ}_{\mathrm{L}})^*(\mathrm{d}P_{\mathrm{L}}/\mathrm{d}\theta)\right)(\mathrm{d}P_l/\mathrm{d}\theta)\sin(\theta)\,\mathrm{d}\theta$$

and with

$$\sum_{\mathrm{L}}(C^{\circ}_{\mathrm{L}}(d_{\mathrm{L}}P/\mathrm{d}\theta)) = \left(\sum_{l''}(q_{\mathrm{l}}, B^{\circ}_{\mathrm{BL},l'}(\mathrm{d}P_{l'}/\mathrm{d}\theta))^*\right)\left(\sum_{l''}((q_{l''}\alpha^{\circ}_{\mathrm{s},l''}(h^1_{l''}(r))'|_{\mathrm{R}}P_{l''}) + (q_{l''}A_{\mathrm{NC}}(j_{l''}(r))'|_{\mathrm{R}}P_{l''}))\right)$$

The h^1_l and the j_l are spherical Hankel and Bessel functions. The constants d°_0 and d°_1 arise in the homogeneous solution to [8], while the remaining terms represent the particular solution due to forcing by the acoustic field variables. The $\hat{\#}u^{\circ}_{\mathrm{r,HY}}$ and $\hat{\#}p^{\circ}_{\mathrm{PH}}$ terms in the sublayer region occur at higher orders in the expansion parameter, $\sqrt{(\delta/\mathrm{Re}_{\mathrm{HY}})}$. Further details can be found in appendix II (available from the Editor). The contribution to $\underline{\mathbf{u}}^{\circ}_{\mathrm{HY}}$ from the sublayer region is completed via multiplication of [9] by EXP. The condition that the velocity field contribution which arises due to forcing by viscous acoustic terms decay at the edge of the sublayer, i.e. as $\zeta \to \infty$, requires that d_0^{o} and d_1^{o} be zero.

Outer host fluid region

There are no sources of vorticity in this region. The solutions are found to be

$$\hat{u}^{\text{o}}_{r,\text{HY}}(r,\theta) = \sum_l (a_2^{\text{o}} r^{-(l+2)} + a_4^{\text{o}} r^{-1} h_l^1(sr)) P_l(\cos(\theta))$$

$$\hat{u}^{\text{o}}_{\theta,\text{HY}}(r,\theta) = \sum_l (l(l+1))^{-1} (\text{d}P_l/\text{d}\theta)(-la_2^{\text{o}} r^{-(l+2)} + a_4^{\text{o}}((l+1)r^{-1}h_l^1(sr) - sh_{l+1}^1(sr)))$$

$$\hat{p}^{\text{o}}_{\text{HY}}(r,\theta) = (\text{Re}_{\text{HY}})^{-1} \sum_l (s^2(l(l+1))^{-1})(-la_2^{\text{o}} r^{-(l+1)}) P_l(\cos(\theta)) \qquad [10]$$

with $s^2 = (-i\text{Re}_{\text{HY}})$. The time dependent factor EXP in the above velocity and pressure fields has been suppressed. Note the form of the hydrodynamic solution in the outer region is the same as that in the problem of an unforced oscillating drop in an infinite fluid medium, as investigated previously by Miller & Scriven (1968).

The hydrodynamic field solution of the acoustically forced oscillating drop in the host region has been determined, and both inner and outer layer solutions exhibited. There remain unknown coefficients in the functional form of the solutions. These are to be determined via application of the boundary–interface conditions at the drop–host fluid interface.

The development of the hydrodynamic field solutions in the drop involves an inner region solution, which occurs in a sublayer region of thickness $\sqrt{(\nu_{\text{o}}^{\text{o}}/\omega_{\text{AC}})}$ adjacent to the drop–host interface, and an outer region solution. The expansion of the conservation of mass and momentum equations in the sublayer region of the drop is performed in the stretched variable ξ, with $r = R - \epsilon\xi$. The procedure is the same as that done for the host region flow field, so only the solutions in the drop region are presented.

Inner drop region

$$\hat{u}^{\text{i}}_{\theta \text{l},\text{HY}} = \sum_{\text{l}} (\text{d}P_{\text{l}}/\text{d}\theta)(T1 + T2 + T3)\exp((-(1+i)(\beta/2\alpha)^{1/2})\xi)\, \text{e}^{i(\eta'' - \eta' - \eta_{\text{RSP,l}})} \qquad [11]$$

with

$$T1 = \text{Re}_{\text{HY}}(2\beta/\alpha)^{1/2}(1+i)(C1_l^{\text{i}})\exp((-(1-i)/(\beta/2\alpha)^{1/2})\xi)$$

$$T2 = \text{Re}_{\text{HY}}(2\beta/\alpha)^{1/2}(1-i)(C2_l^{\text{i}})\exp((-(1+i)/(\beta/2\alpha)^{1/2})\xi)$$

and

$$T3 = d_0^{\text{i}} + d_1^{\text{i}}(\xi)$$

and with

$$C1_l^{\text{i}} = ((2l+1)/2l(l+1)) \int_0^{\pi} \left(\sum_{\text{L}} C_{\text{L}}^{\text{i}}(\text{d}P_{\text{L}}/\text{d}\theta)\right)(\text{d}P_l/\text{d}\theta)\sin(\theta)\,\text{d}\theta$$

$$C2_l^{\text{i}} = ((2l+1)/2l(l+1)) \int_0^{\pi} \left(\sum_{\text{L}} (C_{\text{L}}^{\text{i}})^*(\text{d}P_{\text{L}}/\text{d}\theta)\right)(\text{d}P_l/\text{d}\theta)\sin(\theta)\,\text{d}\theta$$

and with

$$\sum_{\text{L}} (C_{\text{L}}^{i}(\text{d}_{\text{L}}P/\text{d}\theta)) = \left(\sum_{\text{l}'} (q_{\text{l}'} A^{\text{i}}_{\text{BL},\text{l}'}(\text{d}P_{\text{l}'}/\text{d}\theta))^*\right)\left(\sum_{\text{l}''} (q_{\text{l}''}\alpha^{\text{i}}_{\text{l}''}(j_{\text{l}''}((c_{\text{o}}^{\text{i}}/c_{\text{o}}^{\text{o}})r))'|_{\text{R}} P_{\text{l}''})\right)$$

The spherical Bessel function is j_l. Constants d_0^{i} and d_l^{i} represent the homogeneous solution contribution, while remaining terms represent the particular solution due to forcing by acoustic field variables. The $\hat{u}^{\text{i}}_{r,\text{HY}}$ and $\hat{p}^{\text{i}}_{\text{HY}}$ terms in the sublayer region occur at higher orders in the expansion parameter, $\sqrt{(\delta/\text{Re}_{\text{HY}})}$. The contribution to $\mathbf{u}^{\text{i}}_{\text{HY}}$ from the sublayer region is completed via multiplication of [11] by EXP. The condition that the velocity field contribution which arises due to forcing by viscous acoustic terms decay at the edge of the sublayer, i.e. as $\xi \to -\infty$, requires that d_0^i and d_1^i be identically zero.

Outer drop region

There are no sources of vorticity in this region. The solutions are found to be

$$\tilde{u}^i_{r,\mathrm{HY}}(r,\theta) = \sum_l (a_1^i r^{(l-1)} + a_3^i r^{-1} j_l(Sr)) P_l(\cos(\theta))$$

$$\tilde{u}^i_{\theta,\mathrm{HY}}(r,\theta) = \sum_l (l(l+1))^{-1} (\mathrm{d}P_l/\mathrm{d}\theta)((l+1)a_1^i r^{(l-1)} + a_3^i((l+1)r^{-1} j_l(Sr) - Sj_{l+1}(Sr)))$$

$$\tilde{p}^i_{\mathrm{HY}}(r,\theta) = (\mathrm{Re}_{\mathrm{HY}})^{-1} \sum_l (S^2(\alpha/l)(a_1^i r^{(l)}) P_l(\cos(\theta))) \qquad [12]$$

with $S^2 = (-i\mathrm{Re}_{\mathrm{HY}}\beta/\alpha)$. The time dependent factor EXP in [12] has been suppressed. Note that the form of the hydrodynamic solution in the outer region of the drop is that of an unforced oscillating drop in an infinite medium, determined by Miller & Scriven (1986).

The hydrodynamic field solution of the acoustically forced oscillating drop in both drop and host regions has been determined, and the inner and outer layer solutions in the respective regions exhibited. There remain unknown coefficients in the functional forms of the solutions for both the inner and outer layer solutions; these are (a_2^o, a_4^o, a_1^i, a_3^i,). Other coefficients which occur in the inner layer solutions in both drop and host regions originate with the acoustic field problem, and are known quantities. The remaining coefficients are determined via an application of the boundary–interface conditions at the drop–host fluid interface.

Application of the conditions at the drop–host fluid interface provides a set of forced simultaneous algebraic equations in the unknown coefficients. The solution of this set via numerical linear algebra techniques completes the hydrodynamic field problem. Physically, conditions are those of: (a) kinematic condition, (b) continuity of velocity across drop–host fluid interface, (c) tangential stress balance across interface and (d) normal stress balance across the interface, which includes the surface tension contribution. Forcing of the set of algebraic equations is due to the radiation pressure vector, which is the projection of the radiation stress tensor upon the drop surface. (For a brief discussion, see appendix IV, available from the Editor.) Thus, the acoustic field enters the hydrodynamic problem in two ways: (1) as a forcing of the hydrodynamic field conservation of momentum equation and (2) via the radiation surface pressure vector in both radial and tangential directions.

The equilibrium interface is that of an oscillating drop, and can be described by

$$\mathrm{Fe} = r - R - \sum_l (\kappa_l P_l(\cos\theta)\mathrm{EXP}) = 0 \qquad [13]$$

where P_l are Legendre polynomials. It is clear that the interface lies in the sublayer region. The coefficient κ_l is another unknown. This results in five linear algebraic equations in five unknowns. (Since the analysis is axisymmetric, the continuity of velocity across the interface condition represents two equations, one in the $\hat{e}_r$ direction and the second in $\hat{e}_\theta$.)

The boundary–interface conditions are non-dimensionalized and expanded in terms of the parameter $\sqrt{(\delta/\mathrm{Re}_{\mathrm{HY}})}$. To lowest order, for each mode l, the conditions are listed below, with explicit time dependence suppressed. The boundary conditions are applied on $r = R$, $\zeta = 0$, $\xi = 0$. (See appendix V, available from the Editor, for details.)

Continuity of radial velocity

$$R^l a_1^i + j_l(\mathrm{SR})a_3^i - R^{-l-1}a_2^o - h_l^1(sR)a_4^o = 0 \qquad [14]$$

Continuity of tangential velocity

$$-lR^{-l-1}a_2^o + ((l+1)h_l^1(sR) - sRh_{l+1}^1(sR))a_4^o - (l+1)R^l a_1^i - ((l+1)j_l(SR) - SRj_{l+1}(SR))a_3^i = 0 \qquad [15]$$

Kinematic condition

$$-i\kappa_l + R^{-l-2}a_2^{\rm o} + R^{-1}h_l^1(sR)a_4^{\rm o} = 0 \qquad [16]$$

It is noted that the three above conditions involve only velocity field contributions which are solutions in the outer regions (both of the drop and host fluids); because the streaming field (in the sublayer regions) contributions to the hydrodynamic field occur at next order in the expansion parameter.

Tangential stress balance

$$\begin{aligned}&\alpha(l(l+1))^{-1}((2(l^2-1)R^{-2}-S^2)j_l(SR)+2SR^{-1}j_{l+1}(SR))a_3^{\rm i}+\alpha R^{l-2}(2(l-1)/l)a_1^{\rm i}\\ &+\alpha(2\mathrm{Re}_{\rm HY})(\beta/\alpha)(C1_l^{\rm i}+C2_l^{\rm i})e^{i(\eta''-\eta'-\eta_{\rm RSP,l})}-(2(l+2)/(l+1))R^{-l-3}a_2^{\rm o}\\ &-(l(1+1)^{-1}((2(l^2-1)R^{-2}-s^2)h_l^1(sR)+2sR^{-1}h_{l+1}^1(sR))a_4^{\rm o}\\ &-(2\,\mathrm{Re}_{\rm HY})(C1_l^{\rm o}+C2_l^{\rm o})e^{i(\eta''-\eta'-\eta_{\rm RSP,l})}\\ =\;&\mathrm{Re}_{\rm HY}(2l+1)(2l(l+1))^{-1}\int_{\rm o}^{\pi}\langle(\overline{pr})_\theta^{\rm TANG}\rangle(\mathrm{d}P_l/\mathrm{d}\theta)\sin(\theta)\,\mathrm{d}\theta\times e^{-i\eta_{\rm RSP,l}}\end{aligned} \qquad [17]$$

Note that $\langle(\overline{pr})_\theta^{\rm TANG}\rangle$ refers to the time averaged component of the radiation pressure vector in the $\hat{e}_\theta$ directions; with the time dependence factored out of both sides. The term $e^{i(\eta''-\eta'-\eta_{\rm RSP,l})}$ represents the phase difference between the response of the hydrodynamic field and the imposed forcing. The third and sixth terms on the left hand side are the result of velocity field contributions which arise in the sublayer due to viscous acoustic source terms in the hydrodynamic field equations.

Normal stress balance

$$\begin{aligned}&-(\alpha/\mathrm{Re}_{\rm HY})(S^2l^{-1}R^l-2(l-1)R^{l-2})a_1^{\rm i}\\ &-(1/\mathrm{Re}_{\rm HY})(s^2(l+1)^{-1}R^{-l-1}-2(l+2)R^{-l-3})a_2^{\rm o}\\ &+(\alpha/\mathrm{Re}_{\rm HY})(2R^{-2})((l-1)j_l(SR)-SRj_{l+1}(SR))a_3^{\rm i}\\ &-(1/\mathrm{Re}_{\rm HY})(2R^{-2})((l-1)h_l^1(sR)-sRh_{l+1}^1(sR))a_4^{\rm o}-(GR^{-2})(l+2)(l-1)\kappa_l\\ =\;&((2l+1)/2)\left(\int_o^{\pi}(\langle\overline{pr}^{\rm RADIAL}\rangle)P_l(\cos(\theta))\sin\theta\,\mathrm{d}\theta\right)\times e^{-i\eta_{\rm RSP,l}}\end{aligned} \qquad [18]$$

with $\langle\overline{pr}^{\rm RADIAL}\rangle$ the time-averaged component of the radiation pressure vector in the $\hat{e}_r$ direction; with time dependence factored out of both sides. The factor $G=(\sigma/\rho_0^{\rm o}d^3\omega_{\rm D}^2)$, with σ denoting the surface tension at the drop–host fluid interface.

Clearly, radiation pressure vector terms, which are due to the projections of the radiation stress tensor upon the drop–host interface, and which appear as boundary terms; contribute to forcing in both tangential and radial directions. These contributions appear on the right hand sides of [17] and [18], respectively. Additional terms which arise from the acoustic field appear only in the tangential stress balance equation. That is, the hydrodynamic field contribution from the sublayer region appears only in the tangential stress balance equation, via the third and sixth terms on the left-hand side. This contribution would not exist if the acoustic field had been taken as irrotational and inviscid. Note that these terms do not involve any unknown quantities. Calculation of the integral in [17] yields

$$2((2l(l+1))/(2l+1))(\beta(C1_l^{\rm i}+c2_l^{\rm i})-C1_l^{\rm o}-c2_l^{\rm o})\times e^{i(\eta''-\eta')}$$

Use of this expression in [17] results in a balance of the contributions due to the acoustic source terms in the hydrodynamic field. That is, although there remains the contribution to the hydrodynamic field from the acoustic sources in the velocity field itself, this contribution cancels the contribution due to the tangential component of the radiation pressure vector in the boundary conditions. Thus, no boundary terms involving tangential radiation pressure ultimately contribute.

Therefore, the remaining non-homogeneous term in the system of algebraic equations is due to the radial component of the radiation pressure vector. The linear system of unknowns

$(a_1^i, a_3^i, a_2^o, a_4^o, \kappa_l)$ can be solved by use of numerical linear algebra techniques. (See appendix VI, available from the Editor, for details.)

4. DISCUSSION

Numerous non-dimensional parameters appear in the hydrodynamic field problem. To place the results in the context of physical scales, it is useful to consider the range of physical parameter values, which are taken from Marston (1980), Marston & Apfel (1980) and Marston & Goosby (1985). Typical values of c_o^o and ω_{AC} are 10^5 cm/s and $(2\pi)200$ kHz, respectively. For values of ν_o^o ranging from 0.01 to 10.0 $\mathrm{cm^2/s}$, corresponding values of $\mathrm{Re_{AC}}$ range from 10^6 to 10^3. Values of ϵ are equal to the inverse of the square root of $\mathrm{Re_{AC}}$, and are clearly small. For nominal values of density and surface tension, for the $l=2$ mode of oscillation, ω_D values range from roughly 17 kHz for a 0.1 mm radius drop to 47 Hz for a 0.5 mm radius drop. Corresponding values of δ range from 0.01 to 0.00004. Taking $\mathrm{Re_{HY}}$ as order one, the streaming flow in the hydrodynamic field which is induced by the acoustic field quantities extends over a distance of 0.0003–0.002 cm, measured from the drop–host interface.

Although the experiments which have been done have not had $\mathrm{Re_{HY}}$ at order one, a qualitative finding of the experiments was that the tangential radiation pressure vector term is not significant. (The Reynolds numbers in the experiments were generally larger than that of the present analytical study.) The analytical results of this work exhibit a mechanism for why this can be the case in spite of what would be one's physical intuition to the contrary, since it is clear that acoustic sources of vorticity do contribute to the tangential radiation pressure vector.

It is noted that if the fluids involved in the analysis were to be taken to be very, very viscous, to the extent that the parameter $\mathrm{Re_{HY}}$ were to become quite small, the $(\epsilon, \delta, \mathrm{Re_{HY}})$ balance would no longer hold, and the analysis in general would not apply.

Finally, a discussion of this work in the context of flows generated by a solid body in translational motion in an unbounded viscous fluid is in order. This situation is in contrast to the present case, in which the fluid drop is undergoing shape oscillations (no translational motion). Also, an interface between the drop–host fluids is present, rather than the surface of a solid body. The work of Riley (1967) reprised and extended much of the solid body effort, and will be used as a basis for comparison.

The parameters R, Rs, ϵ and M occur in Riley (1967), and will be denoted herein with an r subscript. The Reynolds number $\mathrm{R_r} = \epsilon_r M_r^2$ may be considered a conventional Reynolds number. The parameters ϵ_r and M_r can be viewed as the ratio of the oscillation amplitude to a typical (solid body) length scale and as the ratio of the typical length scale to a viscous length scale, respectively. However, $M_r^2 = \omega_r d_r^2/\nu_r$, which also can be viewed as a Reynolds number in which the motion is oscillatory. A streaming Reynolds number $\mathrm{Rs_r} = \epsilon_r^2 M_r^2$ is defined, also.

Since the work of Riley (1967) was time independent, the comparison made herein applies to the static streaming field, which is discussed in appendix III (available from the Editor). Formally, ϵ_r corresponds to δ and M_r^2 to $\mathrm{Re_{AC}}$. However, δ is the ratio of time scales in the present problem. Define the parameter $\mathrm{Rs} = \delta^2 \mathrm{Re_{AC}} = \delta\ \mathrm{Re_{HY}}$. In the present investigation, $\delta \ll 1$ and $\mathrm{Re_{HY}}$ is order one. Therefore, $\mathrm{Rs} \ll 1$. Clearly, $\mathrm{Re_{AC}} \gg 1$ in the present study. For the case of $M_r^2 \gg 1$, $\mathrm{Rs_r} \ll 1$, and $\mathrm{R_r}$ at order one, Riley (1967) finds the flow outside of a shear layer to be Stokes-like. Continuing the formal comparison, this case would correspond to $\mathrm{Re_{AC}} \gg 1$, $\mathrm{Re_{HY}}$ order one, and $\mathrm{Rs} \ll 1$, which is precisely the case in this investigation. Moreover, the flow obtained in the time-independent case (appendix III, from the Editor) outside the shear layer is Stokes flow.

5. CONCLUSIONS

The hydrodynamic field which exists as a result of droplet forcing by frequency modulated acoustic waves has been investigated analytically. A formal expansion procedure has produced a hierarchy of equations in the order parameter δ, which physically represents the ratio of acoustic and hydrodynamic time scales. At lowest order in δ, this yields equations governing the acoustic field. At next order, those governing the hydrodynamic field are obtained. Through the expansion procedure, the hydrodynamic field is related to the acoustic field from which it is generated in a rational manner.

The hydrodynamic field was found to oscillate on a time scale slower than that of the acoustic time scale, with the hydrodynamic time scale corresponding to the inverse of the beat frequency for the modulated acoustic field. The hydrodynamic field was solved in the frame of this new time scale.

The solution of the hydrodynamic problem revealed a structure in both the drop and host fluids, with an inner region in both fluids existing near the interface in which a streaming velocity field due to acoustic forcing occurs. Any investigations of processes which occur near the interface of the oscillating drop would have to be cognizant of this.

The boundary–interface forcing in the tangential stress balance which arose as a result of the radiation pressure was canceled by contributions arising from the velocity streaming terms. Thus, radiation pressure forcing which actually appears in the boundary–interface conditions is only that of the radial component.

Finally, the relationship of this work to flows obtained in previous work involving the periodic translational motion of a solid body in an unbounded viscous fluid has been elucidated.

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